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# Investigating the properties of mixed finite elements

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**Abstract**

The paper considers quadrature-cubic interpolation in problems of restoration of functions of two arguments. The properties of the most common finite elements Q12 (Lagrange version) and Q10 (Serendipity version) are investigated as finite elements. For more than fifty years, researchers of the finite element method have known the procedure of converting the Lagrangian model to the serendipity model. But, as is known, not all results when using this procedure satisfy users, especially proponents of physical interpretation. We are talking about the magnitude of the nodal loads of the uniform mass force of the serendipity finite element. Thus, if we consider the finite element Q10, it receives the physical inadequacy of the "spectrum" as an inheritance from the "parent" pair of Lagrangian finite elements Q8 and Q12. In Pascal's scheme, there is also a hidden connection between the finite element Q10 and the finite element Lagrangian Q12. The analysis of the inherited properties suggests that it is fundamentally possible for a Q10 substitute-base to exist in nature with the same local and integral characteristics. It turns out that the search for such a base goes beyond the capabilities of traditional modeling methods. An alternative substitute-basis Q10 was found by nonmatrix condensation of the prototype of element Q10, i.e., by using the Lagrangian model Q12. The universal nature of the non-matrix transformation of Q12 into Q10 opens up the possibility of designing a model series of mixed finite elements with physically adequate spectra of nodal loads.



*Keywords:* Serendipity finite elements; Lagrange finite elements; physical inadequacy of standard serendipitous finite elements

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## 1. Introduction

The construction of bases of mixed finite elements (FE) is a pressing concern in the problems of orthotropic fields study. We consider this problem on a specific example, when in the direction of the axis $OX$ the field changes according to the law of the cubic parabola, and in the direction of the axis $OY$ - according to the law of the quadratic parabola. Element $Q10$ is a mixture of elements $Q8$ and $Q12$. We are talking about serendipity finite element (SFE), in Lagrange finite elements (LFE) such a problem does not look complicated and has a single solution. Serendipity approximations did not stop at the standard models of Ergatoudis, Irons, and Zienkiewicz (1968). The constructive theory of SFE approximation has been systematically enriching the library of $Q8$ and $Q12$ elements for almost 40 years. Mixed models are on the agenda.

The most popular $Q8$ and $Q12$ were obtained by ingenious selection in 1968 [1]. This was the first successful attempt to eliminate unwanted internal nodes in $Q9$ and $Q16$ LFE. Portraits of zero-level lines of elements $Q8$ and $Q12$ indicate that the history of SFE began with geometric construction. This result was later confirmed by the method of matrix algebra [2] and the non-matrix method of Taylor [3]. The popularity of $Q8$ and $Q12$ models stimulates the search for a square-cubic $Q10$. In the LFE class a similar model was obtained by Yu.I. Nemchinov [4]. It is $Q12$ element, which is a mixture of $Q9$ and $Q16$. It is worth noting that information on mixed models is very limited. We can call the linear-quadratic model [5, 6] and the linear-cubic model [7]. The authors [8] warned about specific difficulties in the problems of interpolation of functions of two arguments even before the appearance of SFE. With the appearance of SFE there have been more questions than answers, possibly because SFE do not have one-dimensional analogues. If we remove the restriction on the degree of the interpolation polynomial as it is done in [9], we can obtain alternative bases of SFE [10, 11, 12, 13]. The results below confirm that the search for alternatives continues.

## 2. Main results of the study

### *2.1. Purpose of The Study*

To construct a standard base of mixed $Q10$ SFE, which implements quadratic-cubic interpolation. To justify what and how $Q10$ inherits from the "parent" pair $Q8$ and $Q12$. To find the substitute-base of $Q10$ element based on hereditary properties. To show that the method of non-matrix condensation generates $Q10$ models with physically adequate spectra of nodal loads.

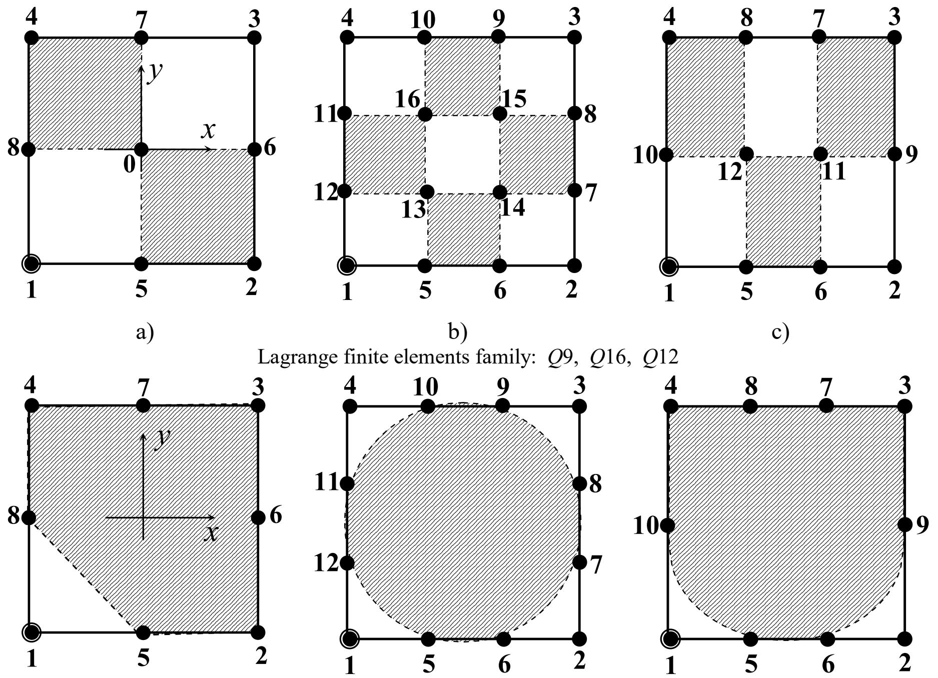
### *2.2. Research Material*

Let the carrier of finite functions be the canonical square: $|x| \le 1, |y| \le 1$. Standard SFE $Q8$ and $Q12$ are used as ingredients for the formation of $Q10$. Similarly, $Q12$ is formed for LFE $Q9$ and $Q16$. Schematically the hybridization process is shown on the example of angular surfaces $N_1(x, y)$ (fig. 1). Cognitive and graphic analysis of fig. 1 helps to learn what and how mixed models (fig. 1, c, f) inherit from their “parent” pair of models. There is some information about the $Q12$ LFE model (fig. 1, c) in the book [4]. Little-known SFE model $Q10$ is in the focus of our attention (fig. 1, f).

Let us remind what the basic functions of known models look like:

*Q9* LFE (fig. 1, a):

$$N_0(x, y) = (1 - x^2)(1 - y^2);$$
$$N_1(x, y) = \frac{1}{4}(1 - x)(1 - y)xy; \text{ similarly } N_i(x, y) \text{ for } i = 2, 3, 4; \qquad (1)$$
$$N_5(x, y) = \frac{1}{2}(1 - x^2)(y - 1)y; \text{ similarly } N_i(x, y) \text{ for } i = 6, 7, 8.$$




a) b) c)

Lagrange finite elements family: $Q9$, $Q16$, $Q12$

d) e) f)

Serendipity finite elements family: $Q8$, $Q12$, $Q10$

Fig. 1. Zero level lines of angular surfaces $N_1(x,y)$

(areas of negative values $N_1(x,y)$ are hashed)

*Q16* LFE (fig. 1, b):

$$N_1(x,y)=\frac{1}{256}(1-x)(1-y)(1-9x^2)(1-9y^2);\ \text{similarly } N_i(x,y) \text{ for } i=2,3,4;$$
$$N_5(x,y)=\frac{9}{256}(1-x^2)(1-3x)(1-y)(9y^2-1);\ \text{similarly } N_i(x,y) \text{ for } i=\overline{6,12}; \quad (2)$$
$$N_{13}(x,y)=\frac{81}{256}(1-x^2)(1-y^2)(1-3x)(1-3y);\ \text{similarly } N_i(x,y) \text{ for } i=14,15,16.$$

*Q12* LFE (fig. 1, c):

$$N_1(x,y)=\frac{1}{32}(1-x)(1-y)(1-9x^2)y;\ \text{similarly } N_i(x,y) \text{ for } i=2,3,4;$$
$$N_5(x,y)=\frac{9}{32}(1-x^2)(1-3x)(y-1)y;\ \text{similarly } N_i(x,y) \text{ for } i=6,7,8;$$
$$N_9(x,y)=\frac{1}{16}(1+x)(9x^2-1)(1-y^2);\ \text{similarly } N_i(x,y) \text{ for } i=10; \quad (3)$$
$$N_{12}(x,y)=\frac{9}{16}(1-x^2)(1-3x)(1-y^2);\ \text{similarly } N_i(x,y) \text{ for } i=11.$$

*Q8* SFE (standard) (fig. 1, d):

$$N_1(x,y)=\frac{1}{4}(1-x)(1-y)(-1-x-y);\ \text{similarly } N_i(x,y) \text{ for } i=2,3,4;$$
$$N_5(x,y)=\frac{1}{2}(1-x^2)(1-y);\ \text{similarly } N_i(x,y) \text{ for } i=6,7,8; \quad (4)$$

*Q12* SFE (standard) (fig. 1, e):

$$N_1(x,y)=\frac{1}{32}(1-x)(1-y)\left(9\left(x^2+y^2\right)-10\right);\text{ similarly } N_i(x,y) \text{ for } i=2,3,4;$$
$$N_5(x,y)=\frac{9}{32}\left(1-x^2\right)(1-3x)(1-y);\text{ similarly } N_i(x,y) \text{ for } i=\overline{6,12}; \quad (5)$$

It remains to find a standard base for a $Q10$ mixed model (fig. 1, f).

Let us remind that the base functions $N_i(x,y)$ must satisfy the conditions of the Lagrange interpolation hypothesis:

$$N_i(x_k,y_k)=\begin{cases}1, & i=k,\\ 0, & i\neq k;\end{cases} \qquad \sum_{i=1}^{M}N_i(x,y)=1 \quad (6)$$

where $i$ is number of function, $k$ is number of node, $M$ is amount of nodes.

Nodal loads $\gamma_i$ from a singular mass force are determined by the Newton-Cotes formula:

$$\gamma_i=\frac{1}{S}\iint_D N_i(x,y)\,dxdy,\text{ where } S \text{ is FE area.} \quad (7)$$

Today there are several methods for constructing the base for SFE: the inverse matrix method [2, 4, 5], the Taylor's method [3, 14, 15], matrix condensation (Jordan, 1970), non-matrix condensation [16], direct geometric construction [17, 18, 19, 20]. First let us figure out what the inverse matrix method gives for $Q10$. The interpolation polynomial is built on the basis of Pascal's scheme (fig. 2, a):

$$f(x,y)=\alpha_1+\alpha_2x+\alpha_3y+\alpha_4x^2+\alpha_5xy+\alpha_6y^2+\alpha_7x^3+\alpha_8x^2y+\alpha_9xy^2+\alpha_{10}\begin{cases}x^3y\\ x^2y^2\\ x^3y^2\end{cases}. \quad (8)$$

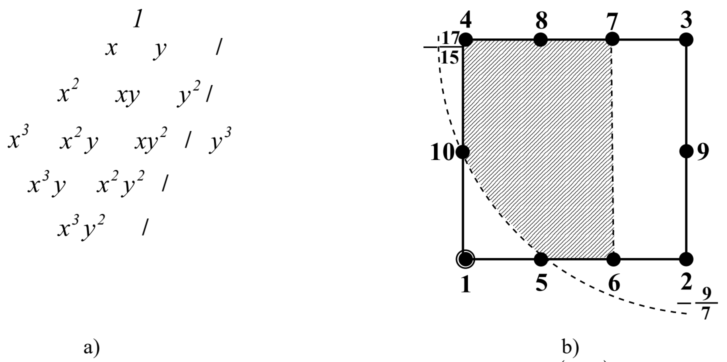



Fig. 2. (a) Pascal's scheme for $Q10$; (b) zero level lines $N_1(x,y)$ base substitute $Q10$

The aim is to solve a system of linear algebraic equations with a 10x10 matrix. It seems that you can get three bases depending on the choice of the tenth monomial (fig. 2, a).

In fact, only one monomial $x^3y$ works. In the other two cases the determinant of the matrix is zero, although the interpolation remains quadratically cubic. The situation resembles Bertrand's paradox from probability theory: the answer depends on the method. Similar cases are found in the theory of interpolation [8]. Below we will try non-matrix condensation [16], and now we are giving the basis obtained by the matrix method.

*Q10* SFE (standard) (fig. 1, f):

$$N_1(x,y)=\frac{1}{32}(1-x)(1-y)\left(9x^2-8y-9\right),\text{ similarly } N_i(x,y) \text{ for } i=2,3,4;$$
$$N_5(x,y)=\frac{9}{32}\left(1-x^2\right)(1-3x)(1-y);\text{ similarly } N_i(x,y) \text{ for } i=6,7,8; \quad (9)$$
$$N_{10}(x,y)=\frac{1}{2}(1-x)\left(1-y^2\right);\text{ similarly } N_9(x,y).$$

Here are some local and integral characteristics of the models to understand what and how mixed models Q12 and (most importantly) Q10 inherit. The main numerical characteristics are nodal loads $\gamma_i$ and barycentric applicates $N_i(0,0)$. Applicates in the center of the carrier make it possible to test the second condition of hypothesis easily and quickly (6).

*Q9* LFE:

$$\gamma_i = \frac{1}{36}, \quad i=\overline{1,4}; \quad \gamma_i = \frac{4}{36}, \quad i=\overline{5,8}; \quad \gamma_0 = \frac{16}{36}; \; N_i(0,0)=0, \quad i=\overline{1,8}; \quad N_0(0,0)=1. \tag{10}$$

*Q16* LFE:

$$\gamma_i = \frac{1}{64}, \quad i=\overline{1,4}; \quad \gamma_i = \frac{3}{64}, \quad i=\overline{5,12}; \quad \gamma_i = \frac{9}{64}, \quad i=\overline{13,16}; \; N_i(0,0)=\frac{1}{256}, \quad i=\overline{1,4}; \\ N_i(0,0)=-\frac{9}{256}, i=\overline{5,12}; \; N_i(0,0)=\frac{81}{256}, \tag{11}$$

*Q12* LFE:

$$\gamma_i = \frac{1}{48}, \quad i=\overline{1,4}; \quad \gamma_i = \frac{3}{48}, \quad i=\overline{5,8}; \quad \gamma_i = \frac{4}{48}, \quad i=9,10; \quad \gamma_i = \frac{12}{48}, i=11,12; \; N_i(0,0)=0, \\ i=\overline{1,4}; \; N_i(0,0)=0, i=\overline{5,8}; \; N_i(0,0)=-\frac{1}{16}, \quad i=9,10; \quad N_i(0,0)=\frac{9}{16}, \quad i=11,12 . \tag{12}$$

It should be noted that the Q12 mixed model inherited a physically adequate spectrum of nodal loads. This fact fundamentally distinguishes LFE from SFE. The influence of the "parent" pair Q9 and Q16 on the synthetic element Q12 is balanced by the rule of geometric averaging. For example, the total number of nodes $12=\sqrt{9 \cdot 16}$, the number of internal nodes $2=\sqrt{1 \cdot 4}$, the angular load $1/48=\sqrt{(1/36)\cdot(1/64)}$. Portraits of zero level lines of angular surfaces (fig. 1, a, b, c) indicate that the Q12 model inherits a grid from orthogonal lines ("chessboard").

Now let us give similar information about SFE (fig. 1, d, e, f).

*Q8* SFE:

$$\gamma_i = -\frac{1}{12}, \quad i=\overline{1,4}; \quad \gamma_i = \frac{1}{3}, \quad i=\overline{5,8}; \quad N_i(0,0)=-\frac{1}{4}, \quad i=\overline{1,4}; \; N_i(0,0)=\frac{1}{2}, \quad i=\overline{5,8} . \tag{13}$$

*Q12* SFE:

$$\gamma_i = -\frac{1}{8}, \quad i=\overline{1,4}; \quad \gamma_i = \frac{3}{16}, \quad i=\overline{5,12}; \; N_i(0,0)=-\frac{5}{16}, \quad i=\overline{1,4}; \; N_i(0,0)=\frac{9}{32}, \; i=\overline{5,12}. \tag{14}$$

*Q10* SFE:

$$\gamma_i = -\frac{5}{48}, \quad i=\overline{1,4}; \quad \gamma_i = \frac{9}{48}, \quad i=\overline{5,8}; \quad \gamma_i = \frac{16}{48}, \quad i=9,10; \\ N_i(0,0)=-\frac{9}{32}, \quad i=\overline{1,4}; \; N_i(0,0)=\frac{9}{32}, \; i=\overline{5,8}; \quad N_i(0,0)=\frac{1}{2}, \; i=9,10. \tag{15}$$

We call the Q10 model the standard one because it is formed from standard Q8 and Q12. It inherited a physically inadequate spectrum $\{\gamma_i\}$. The influence of the "parent" pair on the synthetic element Q10 is balanced by the rule of arithmetic averaging. For example, the number of nodes $10=(1/2)(8+12)$, angular loads $-(5/48)=(1/2)((-1/12)-(1/8))$, barycentric applicate of the angular surface $-(9/32)=(1/2)((-1/4)-(5/16))$.

We will comment on the appearance of the parabola (fig. 1, f). It is interesting what features of the "parent" pair this second-order curve inherits. Cognitively graphical analysis of portraits (fig. 1, d, e) suggests that the infinite length of the parabola is from a line (fig. 1, d), and the curvature is from a circle (fig. 1, e). Another second-order curve, hyperbola, has the same features. This fact inspires the search for alternative bases for *Q*10. The inexhaustible potential of Pascal's scheme reminds us of the possible existence of alternatives (fig. 2, a). Monomials $x^2y^2$ and $x^3y^2$ are not yet involved. Let us try the method of non-matrix condensation [16]. This is a method of direct transformation of the prototype into an image, i.e., LFE *Q*12 (fig. 1, c) - into SFE *Q*10 (fig. 1, f). Internal surfaces $N_{11}(x,y)$ and $N_{12}(x,y)$ also act as correcting ones for other surfaces. Respectively the loads $\gamma_{11}$ and $\gamma_{12}$ are distributed according to a certain "recipe" between the boundary nodes. For example, it is convenient to distribute $\gamma_{12}$ between nodes 1, 4, 5, 8, 10. New (alternative) bases of the function *Q*10 are linearly combined with the corresponding functions *Q*12 according to a rule:

$$\overline{N_1}(x,y) = N_1(x,y) + \alpha \cdot N_{12}(x,y) \tag{16}$$

New load:

$$\overline{\gamma_1} = \gamma_1 + \alpha \cdot \gamma_{12}. \tag{17}$$

We order $\overline{\gamma_1}$, determine the coefficient $\alpha$ from (17) and find $\overline{N_1}(x,y)$ through formula (16). It is interesting to order the standard spectrum:

$$-\frac{5}{48} = \frac{1}{48} + \alpha \cdot \frac{12}{48}, \quad \alpha = -\frac{1}{2}$$

Therefore,

$$\overline{N_1}(x,y) = \frac{1}{32}(1-x)(1-y)(1-3x)(-9-9x-8y-6xy) \tag{18}$$

As you can see the non-matrix approach changed the appearance of Q10 angular surfaces. Finally, the full potential of Pascal's scheme is used (fig. 2, a), and the expected hyperbola appears in the portrait of the zero level lines (fig. 2, b). We received the base substitute SFE Q10: non-standard base with standard characteristics. It means that there are a lot of such models. By the way, the intermediate functions from the set (9) remain, and the angular functions are determined by weighted averaging (9) and (18). For example, the arithmetic average looks like:

$$\overline{N_1}(x,y) = \frac{1}{32}(1-x)(1-y)\left(18x^2 + 9x^2y + 9xy + 9x - 8y - 9\right) \tag{19}$$

Multitude is also a hereditary property. It is transmitted through the hierarchical forms of monomials of Pascal's scheme.

It is also worth giving an example of the Q10 model with a physically adequate range of nodal loads:

$$\begin{aligned} \overline{N_1}(x,y) &= \frac{1}{32}(1-x)(1-y)\left(1-9x^2\right)\cdot y, \quad \gamma_1 = \frac{1}{48}; \\ \overline{N_5}(x,y) &= \frac{9}{32}\left(1-x^2\right)(y-1)(3x+y), \quad \gamma_5 = \frac{3}{48}; \\ \overline{N_{10}}(x,y) &= \frac{1}{2}(1-x)\left(1-y^2\right), \quad \gamma_{10} = \frac{16}{48}. \end{aligned} \tag{20}$$

In 1971 O. Zienkiewicz wrote that nodal loads could not be predicted. Today we model them on order [17, 21, 22, 23, 24].

Leading specialists associate further development of the FEM (finite element method) with incompatible FE and non-polynomial functions of the form. For a long time the use of incompatible FEs was considered incorrect, although engineering-oriented users boldly used incompatible FEs and often obtained quite decent results. After the appearance of the piecewise testing procedure the attitude towards incompatible FEs changed for the better. However, the list of incompatible FEs that can withstand piecewise testing is growing very slowly. In the theory of serendipity FEs, in particular, mixed ones, the phenomenon of incompatibility has not been studied yet. Hereafter we will try to consider this phenomenon through the prism of heredity. Academic interest and practical need stimulate the search for and testing of incompatible mixed FEs. On the example of Q10 mixed element we will show two models of incompatible bases that can withstand piecewise testing by Irons-Razzak [18].

The mixed models $Q10$ considered above (fig. 1) are formed under the balanced influence of the "parent" pair $Q8$ and $Q12$. It is interesting to analyze the consequences of excessive influence of one of the elements ($Q8$ or $Q12$) over $Q10$. For example, to maximize the effect of $Q8$, you can deliberately include another line in the portrait of the zero level lines of the angular surface $N_1(x,y)$ of the element $Q10$ (fig. 3, a). If you use an ellipse passing through all intermediate nodes $Q10$ it will indicate an excessive influence of the $Q12$ model circle (fig. 3, b).

It is interesting that all intermediate surfaces do not differ from standard ones (model with a parabola, fig. 1, f). All numerical characteristics (integral and local) are preserved. The angular surfaces of the models (fig. 3) now look like this:

a) model with two symmetrical lines

$$N_1(x,y) = \frac{1}{32}(1-x)(1-y)(3x+2y+3)(3x-2y-3); \tag{21}$$

b) model with ellipse

$$N_1(x,y)=\frac{1}{32}(1-x)(1-y)(9x^2+8y^2-9) \quad (22)$$

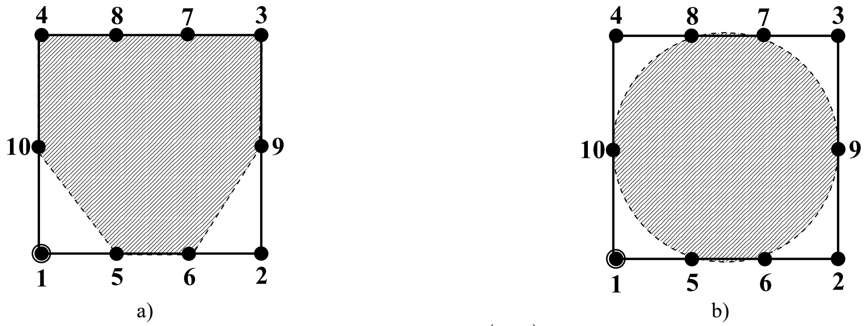



Fig.3 Zero level lines of incompatible surfaces $N_1(x,y)$: (a) stereotype $Q8$; (b) stereotype $Q12$ (areas of negative values $N_1(x,y)$ are hashed)

It is these surfaces $N_i(x,y)$, $i=\overline{1,4}$ that break the interelement continuity. However, incompatibility is observed only in the areas of boundary $x=\pm 1$. Such models can be called "semi-compatible". This term appeared in Kyiv FEM school. The cause of bifurcations emergence on the boundaries $x=\pm 1$ is related to an unwanted monomial $y^3$ (see Pascal's scheme). Unwanted does not mean forbidden. If the model can withstand piecewise testing, it is recommended for use [20]. Therefore, we will study the nature of the behavior of the surface $N_1(x,y)$ on the boundary $x=-1$.

The angular surface of the standard model on the boundary section $x=-1, \quad -1 \le y \le 1$ changes according to the law of the square parabola:

$$N_1(-1,y)=\frac{1}{2}(y-1)\cdot y \quad (23)$$

The angular surface of the model with two lines (fig. 3, a) changes according to the law of the cubic parabola:

$$N_1(-1,y)=\frac{1}{4}(1-y)\cdot y\cdot(-3-y) \quad (24)$$

The angular surface of the model with an ellipse on the specified section of the boundary changes according to the law of the cubic parabola:

$$N_1(-1,y)=\frac{1}{2}(1-y)\cdot y^2 \quad (25)$$

At the control section of the boundary there is a gap of the first kind. The magnitude of the jump for the first model (fig. 3, a) looks like:

$$\varphi(y)=\frac{1}{4}(y-y^3) \quad (26)$$

for the other model (fig. 3, b):

$$\varphi(y)=\frac{1}{2}(y^3-y) \quad (27)$$

As you can see, in both cases the gap has the same nature. Applicates differ only in the sign and the magnitude of the deviation from zero. In figure 4 both curves $\varphi(y)$ are shown, the graph of the second model (27) is shown dotted.

Two graphics in one figure give a simple image not only illustrative properties, but also cognitive. Looking at fig. 4 we make certain that both models can withstand piecewise testing not only by the Irons-Razzak criterion, but also by the Patterson criterion. Here we are dealing with orthogonal polynomials. Extreme values of deviation from zero are reached at points $y=\pm(1/\sqrt{3})$. It is at these points that the nodes of the Gauss-Legendre quadrature are located. In such cases, it is said that the mathematical model has side adequacy. In terms of the speed of convergence of the finite elemental solution to the exact one, the first model is more efficient.

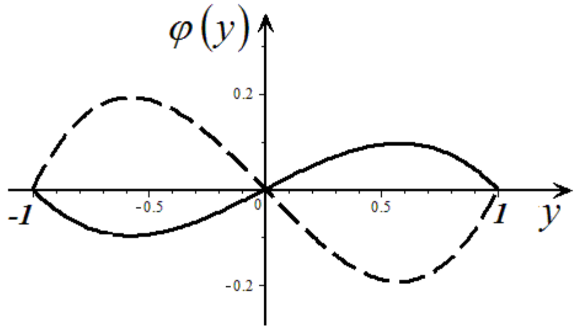



Fig. 4. Change of jump $\varphi(y)$ at the gap boundary $x=-1$

Now that the incompatibility has receded into the background, we will point to another advantage of incompatible models (fig. 3). Simplicity and clarity of portraits of zero level lines open the way to direct geometric modeling of serendipity elements. And reflections over fig. 4 stimulate the attempt to "go to zero". It seems that it is possible to choose the weighting coefficients so that the weighted averaging of the two incompatible surfaces (21) and (22) will lead to a compatible surface $N_1(x,y)$. Judging from fig. 4 and formulas (26) and (27), the coefficients should be 2:1. Let us denote the angular function of the first model by $N_1^{(a)}(x,y)$ and give it a coefficient $2/3$, we will denote the second function by $N_1^{(b)}(x,y)$ with a coefficient $1/3$. Hereafter we will find the weighted average of the surfaces (21) and (22):

$$N_1(x,y)=\frac{2}{3}\cdot N_1^{(a)}(x,y)+\frac{1}{3}\cdot N_1^{(b)}(x,y) \tag{28}$$

Skipping the intermediate transformations, we obtain:

$$N_1(x,y)=\frac{1}{32}(1-x)(1-y)\left(9x^2-8y-9\right) \tag{29}$$

This is a standard angular surface of the element Q10, which is compatible. Yet, this time the model with a parabola comes from another "parent" pair (fig. 3). However, heredity is noticeable. The infinite length of the parabola is inherited from the lines (fig. 3, a), and the curve - from the ellipse (fig. 3, b). Let us recall that the integral and local numerical characteristics of the standard model Q10 are also inherited.

## 3. Conclusions

Mixed models of SFE in terms of hereditary properties are considered for the first time. The example of the Q10 model shows that the result of hybridization is influenced not only by the “parent” pair SFE (Q8 and Q12), but also by the LFE prototype. The cases in which numerical characteristics are insensitive to changes of surface relief attract attention. This example is illustrated by the new model Q10 obtained in the work. The non-matrix condensation procedure generates a lot of mixed Q10 models, including those with physically adequate nodal load spectra. This conclusion applies to all SFEs.

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